Appendix to Tail Risk Interdependence

1. Joint tails and the tail interdependence structure

Let $\mathcal{N} = \{1, ..., n\}$ be a finite set and F a continuous joint CDF (PDF f) of a vector $\mathbf{X} = (X_1, ..., X_n)$ of n random variables with the support on R^n. For the strictly increasing marginal CDF F_i, $i \in \mathcal{N}$, and the nominal level $\alpha \in (0, 1)$, the left and the right tail with the density mass α under F_i are defined, respectively, as

$$T_i^\alpha(-1) = \{x_i \in R : x_i \leq F_i^{-1}(\alpha)\}, \quad T_i^\alpha(1) = \{x_i \in R : x_i \geq F_i^{-1}(1-\alpha)\}.$$

The joint tail (JT) $T_C^\alpha(\mathbf{d}) \subseteq R^n$ in direction $\mathbf{d} \in \{-1, 1\}^n$ for the subset $C \subseteq \mathcal{N}$ and at the nominal level $\alpha \in (0, 1)$ generalizes the unidimensional tails,

$$T_C^\alpha(\mathbf{d}) = \prod_{i \in C} T_i^\alpha(d_i) \times \prod_{i \in \mathcal{N} \setminus C} R \setminus T_i^\alpha(d_i),$$

i.e., $T_C^\alpha(\mathbf{d})$ is the Cartesian product of univariate tails and their complements. We refer to upper (or positive) JTs when $\mathbf{d} = \mathbf{1} = (1, ..., 1)$ and to lower (or negative) JTs when $\mathbf{d} = -\mathbf{1}$. Otherwise, we call the JTs mixed. Importantly for our purposes, the joint tails $T_C^\alpha(\mathbf{d})$ and $T_B^\alpha(\mathbf{d})$ are disjoint if $C \neq B$. Therefore, the superset $\mathcal{T}^\alpha(\mathbf{d}) = \{T_C^\alpha(\mathbf{d}) : C \subseteq \mathcal{N}\}$ partitions the outcome space into 2^n (the number of all subsets of $\mathcal{N}$) regions.

For a partition $\mathcal{T}^\alpha(\mathbf{d})$ and a joint PDF f, we define the *tail interdependence structure* (TIS) $\mathbf{u}^\alpha(f, \mathbf{d}) = (u_C^\alpha(f, \mathbf{d}))_{C \subseteq \mathcal{N}}$ as an 2^n-dimensional vector, where $u_C^\alpha(f, \mathbf{d})$ is the probability mass of the JT $T_C^\alpha(\mathbf{d})$ under f. When there is no risk of confusion, we write $\mathcal{T}^\alpha$, T_C^α and $\mathbf{u}^\alpha$ instead of $\mathcal{T}^\alpha(\mathbf{d})$, $T_C^\alpha(\mathbf{d})$ and $\mathbf{u}^\alpha(f, \mathbf{d})$, respectively. Clearly, $\mathbf{u}^\alpha$ is a (discrete) PDF as $\mathcal{T}^\alpha$ is a partition of the sample space.

1.1. Definition and Properties of the CTI

The interdependence of the JTs captured by the TIS $\mathbf{u}^\alpha$ is fully defined by the *multi-information* (MI) (Cover and Thomas, 2006),

$$D(\mathbf{u}^\alpha||\boldsymbol{\pi}^\alpha) = \sum_{C\subseteq\mathcal{N}} u_C^\alpha \ln \frac{u_C^\alpha}{\pi_C^\alpha}, \tag{1}$$

where $\boldsymbol{\pi}^\alpha = (\pi_C^\alpha)_{C\subseteq\mathcal{N}}$ is the corresponding TIS under tail independence, $\pi_C^\alpha = \alpha^{\#C}(1-\alpha)^{n-\#C}$ is the probability of the JT T_C^α under tail independence (computed as the product of marginal probabilities of $\#C$ exceedances and $n - \#C$ non-exceedances) and $\#C$ is the cardinality of set C. Note that $D(\mathbf{u}^\alpha||\boldsymbol{\pi}^\alpha)$ is well-defined as $\pi_C^\alpha > 0$ for all $\alpha \in (0,1)$ and $C \subseteq \mathcal{N}$. We normalize MI to obtain the *coefficient of tail interdependence* (CTI),

$$\kappa(\mathbf{u}^\alpha) = \frac{D(\mathbf{u}^\alpha||\boldsymbol{\pi}^\alpha)}{(1-n)\ln \alpha^\alpha(1-\alpha)^{1-\alpha}} \tag{2}$$

where the denominator is a normalization factor derived in the subsection 1.2.

The CTI (2) has the following properties. Firstly, it lies in the unit interval. In particular, $\kappa(\mathbf{u}^\alpha) = 0$ when all exceedances are mutually independent and $\kappa(\mathbf{u}^\alpha) = 1$ in the case of perfect dependence, i.e., when all n variables always exceed together their respective thresholds. Secondly, the CTI is scale invariant under strictly increasing transformations of the underlying variables in $\mathbf{X}$. Specifically, if each $\xi_i(X_i)$ is an increasing and continuous function, then the CTI computed from the transformed variables $\xi(\mathbf{X}) = (\xi_i(X_i))_{i=1,\dots,n}$ is the same as that computed from $\mathbf{X}$. This property follows by the construction of the TIS from the quantiles of the variables in $\mathbf{X}$ as the same events fall into a JT T_C^α under $\mathbf{X}$ and under $\xi(\mathbf{X})$. Further, by the construction of the TIS, the CTI is robust to outliers and is invariant under the permutation of the random variables in $\mathbf{X}$.

In empirical applications, the CTI computation time increases linearly in the product of the sample size T and the dimension n of each observation. However, the CTI will overestimate the tail interdependence if the sample size is below the order of 2^n (the number of JTs for n random variables). Below, we show that the decomposition of the CTI into systemic and residual component circumvents the latter problem.

1.2. Systemic and residual tail interdependence

For a given TIS $\mathbf{u}^\alpha$, we define the *systemic TIS* as the $(n+1)-$dimensional vector $\widetilde{\mathbf{u}}^\alpha = (\widetilde{u}_k^\alpha)_{k=0}^n$ where,

$$\widetilde{u}_k^\alpha = \sum_{C \subseteq \mathcal{N}:\#C=k} u_C^\alpha,$$

is the probability of observing $k = 0, ..., n$ tail events. In the special case of mutual independence of tail events, we denote the systemic TIS by $\widetilde{\boldsymbol{\pi}}^\alpha = (\widetilde{\pi}_k^\alpha)_{k=0}^n$, where

$$\widetilde{\pi}_k^\alpha = \sum_{C \subseteq \mathcal{N}:\#C=k} \alpha^k (1-\alpha)^{n-k} = \binom{n}{k} \alpha^k (1-\alpha)^{n-k}.$$

From the TIS $\mathbf{u}^\alpha$, we compute also the conditional probabilities of JTs $\mathbf{u}^{\alpha,k} = (u_C^\alpha / \widetilde{u}_k^\alpha)_{C \subseteq \mathcal{N}:\#C=k}$ given that k exceedances have occurred. Similarly, we compute the conditional probabilities $\boldsymbol{\pi}^{\alpha,k} = (\pi_C^\alpha / \widetilde{\pi}_k^\alpha)_{C \subseteq \mathcal{N}:\#C=k}$ from the PDF $\boldsymbol{\pi}^\alpha$ for each $k = 0, ..., n$. The total KL divergence $D(\mathbf{u}^\alpha || \boldsymbol{\pi}^\alpha)$ can be decomposed as follows.

$$D(\mathbf{u}^\alpha || \boldsymbol{\pi}^\alpha) = D(\widetilde{\mathbf{u}}^\alpha || \widetilde{\boldsymbol{\pi}}^\alpha) + \sum_{k=0}^n \widetilde{u}_k^\alpha D(\mathbf{u}^{\alpha,k} || \boldsymbol{\pi}^{\alpha,k}). \tag{3}$$

In order to prove the decomposition in (3), we calculate,

$$\begin{aligned} D(\mathbf{u}^\alpha || \boldsymbol{\pi}^\alpha) - D(\widetilde{\mathbf{u}}^\alpha || \widetilde{\boldsymbol{\pi}}^\alpha) &= \sum_{C \subseteq \mathcal{N}} u_C^\alpha \ln \frac{u_C^\alpha}{\pi_C^\alpha} - \sum_{k=0}^n \widetilde{u}_k^\alpha \ln \frac{\widetilde{u}_k^\alpha}{\widetilde{\pi}_k^\alpha} \\ &= \sum_{k=0}^n \sum_{C \subseteq \mathcal{N}:\#C=k} u_C^\alpha \ln \frac{u_C^\alpha}{\pi_C^\alpha} - \sum_{k=0}^n \widetilde{u}_k^\alpha \ln \frac{\widetilde{u}_k^\alpha}{\widetilde{\pi}_k^\alpha} \\ &= \sum_{k=0}^n \widetilde{u}_k^\alpha \left(\sum_{C \subseteq \mathcal{N}:\#C=k} \left(\frac{u_C^\alpha}{\widetilde{u}_k^\alpha} \ln \frac{u_C^\alpha}{\pi_C^\alpha} \right) - \ln \frac{\widetilde{u}_k^\alpha}{\widetilde{\pi}_k^\alpha} \right). \end{aligned} \tag{4}$$

As $\sum_{C\subseteq\mathcal{N}:\#C=k} u^\alpha_C/\widetilde{u}^\alpha_k = 1$, we can write the last expression as,

$$\sum_{k=0}^n \widetilde{u}^\alpha_k \left(\sum_{C\subseteq\mathcal{N}:\#C=k} \left(\frac{u^\alpha_C}{\widetilde{u}^\alpha_k} \ln \frac{u^\alpha_C}{\pi^\alpha_C} - \frac{u^\alpha_C}{\widetilde{u}^\alpha_k} \ln \frac{\widetilde{u}^\alpha_k}{\widetilde{\pi}^\alpha_k} \right) \right) = \tag{5}$$

$$\sum_{k=0}^n \widetilde{u}^\alpha_k \sum_{C\subseteq\mathcal{N}:\#C=k} \left(\frac{u^\alpha_C}{\widetilde{u}^\alpha_k} \ln \frac{u^\alpha_C/\widetilde{u}^\alpha_k}{\pi^\alpha_C/\widetilde{\pi}^\alpha_k} \right) = \sum_{k=0}^n \widetilde{u}^\alpha_k D(\mathbf{u}^{\alpha,k}||\boldsymbol{\pi}^{\alpha,k}),$$

which completes the proof of the decomposition in (3).

The measure $D(\widetilde{\mathbf{u}}^\alpha||\widetilde{\boldsymbol{\pi}}^\alpha)$ quantifies the *systemic* tail interdependence, i.e., the divergence between the distributions $\widetilde{\mathbf{u}}^\alpha$ and $\widetilde{\boldsymbol{\pi}}^\alpha$ of the *total* number of exceedances under $\mathbf{u}^\alpha$ and under $\boldsymbol{\pi}^\alpha$ (i.e., under tail independence), respectively. On the other hand, each KL divergence $D(\mathbf{u}^{\alpha,k}||\boldsymbol{\pi}^{\alpha,k})$ quantifies the conditional interdependence among subsets of variables, given that k exceedances have occurred.

In analogy to the CTI (2), we define the *systemic* and *residual CTIs* as, respectively,

$$\widetilde{\kappa}(\mathbf{u}^\alpha) = \frac{D(\widetilde{\mathbf{u}}^\alpha||\widetilde{\boldsymbol{\pi}}^\alpha)}{(1-n)\ln\alpha^\alpha(1-\alpha)^{1-\alpha}}, \quad \kappa^k(\mathbf{u}^\alpha) = \frac{D(\mathbf{u}^{\alpha,k}||\boldsymbol{\pi}^{\alpha,k})}{(1-n)\ln\alpha^\alpha(1-\alpha)^{1-\alpha}}, \tag{6}$$

and show below that

$$\begin{aligned} \kappa(\mathbf{u}^\alpha) &= \widetilde{\kappa}(\mathbf{u}^\alpha) + \sum_{k=0}^n \widetilde{u}^\alpha_k \kappa^k(\mathbf{u}^\alpha), \\ 0 &\leq \widetilde{\kappa}(\mathbf{u}^\alpha) \leq \kappa(\mathbf{u}^\alpha) \leq 1, \end{aligned} \tag{7}$$

with $\widetilde{\kappa}(\mathbf{u}^\alpha) = \kappa(\mathbf{u}^\alpha) = 0$ in the case of tail independence and $\widetilde{\kappa}(\mathbf{u}^\alpha) = \kappa(\mathbf{u}^\alpha) = 1$ for perfect dependence (i.e., when all exceedances always occur together).

In order to prove (7), we divide both sides of the Equation (3) by $-(n-1)\ln\alpha^\alpha(1-\alpha)^{1-\alpha} > 0$ for $0 < \alpha < 1$, which yields the decomposition of the CTI,

$$\kappa(\mathbf{u}^\alpha) = \widetilde{\kappa}(\mathbf{u}^\alpha) + \sum_{k=0}^n \widetilde{u}^\alpha_k \kappa^k(\mathbf{u}^\alpha).$$

We note that $\kappa(\mathbf{u}^\alpha) \geq \widetilde{\kappa}(\mathbf{u}^\alpha) \geq 0$ follows from the non-negativity of $\widetilde{\kappa}(\mathbf{u}^\alpha)$ and $\kappa^k(\mathbf{u}^\alpha)$ as the KL divergence is always non-negative (Cover and Thomas, 2006). Finally, from the results in Cover and Thomas (2006) it follows that,

$$D(\mathbf{u}^\alpha||\boldsymbol{\pi}^\alpha) = \textstyle\sum_{C\subseteq\mathcal{N}} u_C^\alpha \ln u_C^\alpha - n \ln \alpha^\alpha (1-\alpha)^{1-\alpha};$$
$$-\ln \alpha^\alpha (1-\alpha)^{1-\alpha} \leq -\textstyle\sum_{C\subseteq\mathcal{N}} u_C^\alpha \ln u_C^\alpha \leq -n \ln \alpha^\alpha (1-\alpha)^{1-\alpha},$$

which implies

$$\frac{n \ln \alpha^\alpha (1-\alpha)^{1-\alpha} - \sum_{C\subseteq\mathcal{N}} u_C^\alpha \ln u_C^\alpha}{(n-1) \ln \alpha^\alpha (1-\alpha)^{1-\alpha}} = \frac{-D(\mathbf{u}^\alpha||\boldsymbol{\pi}^\alpha)}{(n-1) \ln \alpha^\alpha (1-\alpha)^{1-\alpha}} = \kappa(\mathbf{u}^\alpha) \in [0,1]. \tag{8}$$

For an empirical TIS $\widehat{\mathbf{u}}^\alpha$, the total divergence $D(\widehat{\mathbf{u}}^\alpha||\boldsymbol{\pi}^\alpha)$, and thus the CTI $\kappa(\widehat{\mathbf{u}}^\alpha)$, is not estimated accurately in high dimensions when there are no sufficient observations in all joint tails. For example, for $n = 30$, there are 2^{30} (over one billion) joint tails, which vastly exceeds the usual sample sizes. However, this is not a problem for the systemic interdependence measure $D(\widehat{\widetilde{\mathbf{u}}}^\alpha||\widetilde{\boldsymbol{\pi}}^\alpha)$ and the systemic CTI $\widetilde{\kappa}(\widehat{\widetilde{\mathbf{u}}}^\alpha)$, as both of which are based on the distribution of the total number (between 0 and n) of exceedances.

1.3. Goodness-of-Fit and Independence tests

Recall that $\mathcal{T}^\alpha$ is a partition of the sample space of the $n-$dimensional random vector $\mathbf{X} = (X_1, ..., X_n)$ into 2^n joint tails and that the TIS $\mathbf{u}^\alpha$ is a PDF over $\mathcal{T}^\alpha$. An empirical TIS $\widehat{\mathbf{u}}^\alpha = (\widehat{u}_C^\alpha)_{C\subseteq\mathcal{N}}$ contains the relative frequencies of observations that fall into the JTs $T_C^\alpha \in \mathcal{T}^\alpha$. We use $\widehat{\mathbf{u}}^\alpha$ to test whether the observed interdependence structure comes from a hypothesized PDF f, which produces the TIS $\mathbf{u}^\alpha$. For this purpose, we compute the KL divergence $D(\widehat{\mathbf{u}}^\alpha||\mathbf{u}^\alpha)$,[1]

$$D(\widehat{\mathbf{u}}^\alpha||\mathbf{u}^\alpha) = \sum_{C\subseteq\mathcal{N}} \widehat{u}^\alpha_C \ln \frac{\widehat{u}^\alpha_C}{u^\alpha_C}. \tag{9}$$

If exceedances are mutually independent under f, this procedure boils down to a test of tail independence. In the latter case, the hypothesized TIS is $\boldsymbol{\pi}^\alpha$ and (9) is proportional to the CTI (2),

$$D(\widehat{\mathbf{u}}^\alpha||\boldsymbol{\pi}^\alpha) = (1-n)\ln \alpha^\alpha (1-\alpha)^{1-\alpha}\kappa(\widehat{\mathbf{u}}^\alpha). \tag{10}$$

Our goodness-of-fit test with the mutual independence test as a special case, is conditional on sufficient statistics estimated from the data (e.g., on the estimates of quantiles in the sample). For the conditional test, the asymptotic distribution of the test statistic $2\cdot T\cdot D(\widehat{\mathbf{u}}^\alpha||\mathbf{u}^\alpha)$, where T is the sample size, follows the χ^2-distribution with d degrees of freedom (e.g., McCullagh, 1986). For the degrees of freedom, we observe that we have 2^n outcomes (JTs) and $n+1$ restrictions on probabilities or frequencies of these outcomes: these probabilities must sum up to one and, moreover,

$$\sum_{C\subseteq\mathcal{N}:i\in C} u^\alpha_C = \sum_{C\subseteq\mathcal{N}:i\in C} \widehat{u}^\alpha_C = \alpha, \quad \forall i = 1,...,n.$$

Therefore, we apply $d = 2^n - n - 1$ degrees of freedom in our goodness-of-fit tests.

Alternatively, we can use the systemic observed TIS $\widehat{\widetilde{\mathbf{u}}}^\alpha$ and the systemic theoretical TIS $\widetilde{\mathbf{u}}^\alpha$ to compute the KL divergence $D(\widehat{\widetilde{\mathbf{u}}}^\alpha||\widetilde{\mathbf{u}}^\alpha)$. In this case, $2\cdot T\cdot D(\widehat{\widetilde{\mathbf{u}}}^\alpha||\widetilde{\mathbf{u}}^\alpha)$ is distributed approximately as χ^2 with $d = n-1$ degrees of freedom as there are

[1] The Goodness-of-Fit and the interdependence symmetry test below can be only conducted when the test statistic is well-defined, i.e., when all denominators in (9) are strictly positive.

$n+1$ outcomes and two restrictions on probabilities of these outcomes,

$$\sum_{k=0}^{n} \widetilde{u}_k^\alpha = 1, \quad and \quad \sum_{k=0}^{n} k\widetilde{u}_k^\alpha = n\alpha.$$

1.4. Interdependence symmetry test

Another interesting question is whether two tail interdependence structures along two directional vectors $\mathbf{d}^+$ and $\mathbf{d}^-$ (e.g., negative and positive tails) are symmetric. Specifically, let $\widehat{\mathbf{u}}^{\alpha+} = (\widehat{u}_C^{\alpha+})_{C\subseteq\mathcal{N}}$ and $\widehat{\mathbf{u}}^{\alpha-} = (\widehat{u}_C^{\alpha-})_{C\subseteq\mathcal{N}}$ be two empirical TISs computed for $\mathbf{d}^+$ and $\mathbf{d}^-$, respectively. Our objective is to test whether $\widehat{\mathbf{u}}^{\alpha+}$ and $\widehat{\mathbf{u}}^{\alpha-}$ were generated by a process with an identical tail interdependence structure. In order to test the null $\mathbf{u}^{\alpha+} = \mathbf{u}^{\alpha-}$, we apply the Kullback–Leibler test statistic,

$$KL^\pm = \sum_{C\subseteq\mathcal{N}} T^+ \widehat{u}_k^{\alpha+} \ln \frac{\widehat{u}_C^{\alpha+}}{\widehat{u}_C^{\alpha}} + \sum_{C\subseteq\mathcal{N}} T^- \widehat{u}_C^{\alpha-} \ln \frac{\widehat{u}_C^{\alpha-}}{\widehat{u}_C^{\alpha}},$$

where,

$$\widehat{u}_C^\alpha = \frac{(T^+\widehat{u}_C^{\alpha+} + T^-\widehat{u}_C^{\alpha-})}{T^+ + T^-},$$

and T^+ (T^-) is the size of the sample from which $\widehat{\mathbf{u}}^{\alpha+}$ ($\widehat{\mathbf{u}}^{\alpha-}$) have been computed. The asymptotic distribution of $2 \cdot KL^\pm$ follows the χ^2-distribution with $2^n - 1$ degrees of freedom (e.g., Quine and Robinson, 1985). We refer to this procedure as the interdependence symmetry test.[2] Alternatively, the statistic $KL^\pm$ can be computed from the systemic TIS, in which case $2 \cdot KL^\pm$ follows the χ^2-distribution with n degrees of freedom.

[2] Alternatively, we can test the null with a generalized version of the Fisher's exact test (Mehta and Hilton, 1993).

2. Further discussion on "Comparison with alternative tail dependence measures"

2.1. Non-parametric bivariate measure proposed by Poon et al (2004)

Poon et al. (2004) propose a bivariate framework to examine tail dependence structure. They distinguish between asymptotic dependence and independence structures by constructing two non-parametric estimators, χ and $\bar{\chi}$, which measure tail dependence when two variables are respectively asymptotically dependent and asymptotically independent. The pair of dependence measures (χ, $\bar{\chi}$) provides all the necessary information for characterising the form and the degree of extremal dependence. A researcher should first test if $\bar{\chi} = 1$. If the hypothesis is not rejected, the variables are asymptotically dependent with tail dependence given by χ and the result presented as (χ,1). Otherwise, the variables are asymptotically independent with tail dependence measured by $\bar{\chi}$ and the result presented as (0, $\bar{\chi}$). For full technical details see Poon et al. (2004).

Since this is a bivariate measure, in the interest of space we only report results for Apple (AA) with the remaining 29 stock returns. The results for the other pairs are qualitatively similar and available upon request. Table 1 reports tail dependence measure $\bar{\chi}$ (pairs displaying asymptotic dependence are marked with an asterisk) whereas Table 2 reports tail dependence measure χ. A couple of observations are in order. At the 5th quantile, only 8 out of the 29 pairs (27% of all pairs) exhibit asymptotic dependence. This result is consistent with of Poon et al (2004) who find that only 15% of their market return pairs display asymptotic dependence. For those pairs exhibiting asymptotic dependence, there is about 30-40% probability that they experience large positive or negative returns, as manifested by the χ coefficient estimated in the range of 0.3 to 0.4. The evidence shows that the left tail is thicker than the right tail. In terms of $\bar{\chi}$, 60% and 90% of the pairs show higher dependence respectively at the 5th and 10th quantiles in the left tail. In addition, more pairs of returns exhibit asymptotic dependence in the left tail than in the right tail at the 5% quantile (eight pairs versus three pairs).

2.2. Parametric high-dimensional factor copula model proposed by Oh and Patton (2017)

Oh and Patton (2017) propose a factor copula model which allows for a fat-tailed common factor to capture the possibility of correlated crashes and an asymmetric distribution to model potential differences in dependence during downturns and upturns. Each of the series may carry different factor coefficients which, using results from extreme value theory, can be used to derive the tail dependence implied by the copula. Since factor models do not have a closed-form likelihood, the model is estimated with the simulated method of moments and is computationally intensive. See Oh and Patton (2017) for full technical details.

We follow Oh and Patton (2017) by estimating a factor skew t-t model, allowing for one common factor in our estimation. We also classify the DJ30 returns into six industries based on their SIC code.[3] Firms with the same first digit in the SIC code share the same factor coefficient. Table 4 reports the estimated model coefficients.

The estimates suggest that the residuals of DJ30 exhibit fat tails (as manifested by the degree of freedom parameter ν of the t-distribution which is close to 2.5) and that the distribution is left-skewed (as manifested by the statistically significant negative coefficient). We also observe large positive coefficients β for each industry, in line with the findings of Oh and Patton (2017).

Following Oh and Patton (2017), we then compute lower and upper tail dependence based on the estimated factor skew t-t model and report the results in Table

[3] Eight firms belong to SIC2 (Manufacturing: food, furniture): DD, JNJ, KO, MO, MRK, PFE, PG, XOM; nine firms belong to SIC3 (Manufacturing: electricity, machinery): AA, BA, CAT, GE, HON, HPQ, INC, MMM, UTX; three firms belong to SIC4 (Transportation, communications): DIS, ATT, VZ; three firms belong to SIC5 (Trade): HD, MCD, WMT; five firms belong to SIC6 (Finance, Insurance): AIG, AXP, C, JPM, TRV; two firms belong to SIC7 (Services): IBM, MSFT.

5. The results suggest a clear asymmetry in dependence: industries display stronger dependence in the lower tail than in the upper tail. For example, in the lower tail, at the 5% quantile the dependence coefficient lies between 0.3 and 0.5, as opposed to the range and 0.2 and 0.4 in the upper tail.

In order to compare the results from Poon et al. (2004) and Oh and Patton (2017) with our measure of tail dependence, we report analogous results using CTI in Table 3 and in Table 6, respectively. We observe that all three models deliver qualitatively similar results. In particular, they show that dependence is stronger in the negative tails, not only across pairs of returns but also across industries, and that it increases when tails become more extreme, i.e. for lower α's.

3. References

Cover, T.M., Thomas, J.A., 2006, Information Theory, 2nd ed., Wiley & Sons.

McCullagh, P., 1986, The Conditional Distribution of Goodness-of-Fit Statistics for Discrete Data, Journal of American Statistical Association 81, 104-107.

Mehta, C.R. and Hilton, J.F. 1993, Exact power of conditional and unconditional tests: Going beyond the 2 times 2 contingency tables. The American Statistician 47, 91-98.

Oh, D. H., Patton, A.J. 2017, Modeling dependence in high dimensions with factor copulas. Journal of Business and Economic Statistics 35, 139-154.

Poon, S-H, Michael R., Jonathan, T. 2004. Extreme value dependence in financial markets: diagnostics, models and financial implications. Review of Financial Studies 17, 581-610.

Quine, M. P. and Robinson, J. 1985, Efficiencies of chi-square and likelihood ratio goodness-of-fit tests. Annals of Statististics 13, 727–742.

Table 1: Tail dependence $\bar{\chi}$ of pairs of DJ30 constituents and Apple with the Poon et al. (2004)

	Left Tail Quantile			Right Tail Quantile		
	5%	10%	20%	20%	10%	5%
AIG	0.71	0.86	0.77	0.58	0.60	0.73
AXP	0.89*	0.81	0.76	0.68	0.74	0.82*
BA	0.80*	0.74	0.69	0.58	0.60	0.63
C	0.90*	0.85*	0.80	0.60	0.60	0.76
CAT	0.80*	0.79	0.79	0.76	0.85*	0.97*
DD	0.89*	0.81	0.81	0.74	0.82	0.87*
DIS	0.68	0.79	0.72	0.61	0.60	0.70
GE	0.86*	0.86*	0.84	0.69	0.72	0.68
HD	0.59	0.60	0.58	0.58	0.58	0.67
HON	0.77	0.77	0.79	0.61	0.69	0.63
HPQ	0.56	0.61	0.65	0.53	0.49	0.44
IBM	0.58	0.56	0.60	0.55	0.54	0.55
INC	0.48	0.53	0.62	0.53	0.54	0.61
JNJ	0.54	0.50	0.53	0.42	0.36	0.50
JPM	0.79*	0.80	0.78	0.61	0.65	0.73
KO	0.57	0.56	0.53	0.46	0.37	0.48)
MCD	0.50	0.47	0.51	0.41	0.42	0.45
MMM	0.69	0.78	0.73	0.70	0.77	0.78
MO	0.42	0.42	0.48	0.42	0.36)	0.35
MRK	0.60	0.57	0.57	0.48	0.51	0.66
MSFT	0.54	0.56	0.60	0.55	0.54	0.63
PFE	0.61	0.57	0.55	0.50	0.51	0.58
PG	0.43	0.45	0.45	0.43	0.38	0.53
ATT	0.61	0.64	0.59	0.53	0.61	0.70
TRV	0.75	0.72	0.66	0.52	0.57	0.62
UTX	0.74	0.73	0.75	0.67	0.68	0.72
VZ	0.64	0.66	0.59	0.52	0.55	0.54
WMT	0.31	0.47	0.46	0.45	0.46	0.48
XOM	0.84*	0.79	0.71	0.66	0.66	0.73

Note: Tail dependence measure $\bar{\chi}$ for each pair of returns with AA assuming that the pair of returns are asymptotically independent. Those pairs indicated with an asterisk * do not reject the null hypothesis of $\bar{\chi} = 1$ and hence are asymptotically dependent; their tail dependence measure is given by χ in Table 2. Otherwise, the pair is considered asymptotically independent and its co-dependence is given by $\bar{\chi}$. The results are estimated with the residuals of the regression for the DJ30 returns on the first FFC factor. Refer to Poon et al. (2004) for details.

Table 2: Tail dependence χ of pairs of DJ30 constituents and Apple with the Poon et al. (2004)

	Left Tail Quantile			Right Tail Quantile		
	5%	10%	20%	20%	10%	5%
AXP	0.35	N.A.	N.A.	N.A.	N.A.	0.33
BA	0.34	N.A.	N.A.	N.A.	N.A.	N.A.
C	0.37	0.40	N.A.	N.A.	N.A.	N.A.
CAT	0.39	N.A.	N.A.	N.A.	0.40	0.37
DD	0.39	N.A.	N.A.	N.A.	N.A.	0.37
GE	0.39	0.41	N.A.	N.A.	N.A.	N.A.
JPM	0.37	N.A.	N.A.	N.A.	N.A.	N.A.
XOM	0.34	N.A.	N.A.	N.A.	N.A.	N.A.

Note: Tail dependence measure χ for each pair of returns with AA assuming that the pair of returns are asymptotically dependent. The results are estimated with the residuals of the regression for the DJ30 returns on the first FFC factor. Refer to Poon et al. (2004) for details.

Table 3: Tail dependence of pairs of DJ30 constituents and Apple with CTI measure

	Left Tail Quantile			Right Tail Quantile		
	5	10%	20%	20%	10%	5%
AIG	0.11	0.1	0.06	0.03	0.05	0.05
AXP	0.11	0.08	0.06	0.05	0.08	0.07
BA	0.11	0.08	0.06	0.04	0.04	0.04
C	0.13	0.1	0.08	0.04	0.05	0.07
CAT	0.12	0.12	0.1	0.08	0.11	0.14
DD	0.15	0.13	0.1	0.08	0.1	0.11
DIS	0.09	0.07	0.06	0.04	0.05	0.06
GE	0.13	0.12	0.09	0.06	0.07	0.09
HD	0.07	0.06	0.04	0.03	0.04	0.06
HON	0.13	0.11	0.08	0.06	0.07	0.07
HPQ	0.07	0.06	0.05	0.04	0.03	0.04
IBM	0.05	0.05	0.05	0.04	0.04	0.04
INC	0.05	0.05	0.05	0.03	0.03	0.05
JNJ	0.03	0.03	0.03	0.01	0.02	0.02
JPM	0.11	0.09	0.07	0.05	0.06	0.08
KO	0.04	0.04	0.03	0.02	0.02	0.02
MCD	0.03	0.03	0.03	0.02	0.02	0.02
MMM	0.1	0.09	0.07	0.06	0.08	0.11
MO	0.03	0.02	0.03	0.02	0.02	0.01
MRK	0.05	0.04	0.03	0.02	0.04	0.04
MSFT	0.05	0.05	0.04	0.03	0.03	0.04
PFE	0.05	0.04	0.03	0.02	0.03	0.02
PG	0.03	0.02	0.03	0.01	0.01	0.02
ATT	0.06	0.05	0.04	0.02	0.04	0.04
TRV	0.06	0.05	0.03	0.02	0.03	0.03
UTX	0.11	0.1	0.09	0.06	0.06	0.09
VZ	0.06	0.05	0.03	0.02	0.02	0.03
WMT	0.01	0.03	0.03	0.02	0.02	0.02
XOM	0.09	0.08	0.06	0.05	0.06	0.07

Note: CTI dependence measure $\bar{\chi}$ for each pair of returns with AA, estimated with the residuals of the regression for the DJ30 returns on the first FFC factor. Refer to the main text for details.

Table 4: Model coefficients of the factor copula model based on Oh and Patton (2017) estimated with DJ30 constituents

	Estimate	Std Error
ν^{-1}	0.41	0.024
λ	-0.05	0.022
β_{SIC2}	0.71	0.015
β_{SIC3}	0.81	0.017
β_{SIC4}	0.86	0.023
β_{SIC5}	0.76	0.020
β_{SIC6}	0.94	0.023
β_{SIC7}	0.87	0.030

Note: This table reports the model estimates of the skew t-t factor copula model proposed by Oh and Patton (2017), using the residuals of the regression for the DJ30 returns on the first FFC factor. The parameter ν^{-1} corresponds to the inverse of the degree of freedom of the t-distribution, λ measures the skewness of the t-distribution (a negative coefficient indicates the distribution skewing to the left), and β refers to the factor coefficients for each industry.

Table 5: Lower\ Upper dependence coefficients among six industries based on Oh and Patton (2017)

Panel A: 5% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.26	0.26	0.26	0.26	0.26
SIC3	0.33		0.33	0.29	0.33	0.33
SIC4	0.33	0.41		0.29	0.36	0.36
SIC5	0.33	0.36	0.36		0.29	0.29
SIC6	0.33	0.41	0.44	0.36		0.27
SIC7	0.33	0.41	0.44	0.36	0.45	

Panel B: 10% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.29	0.29	0.29	0.29	0.29
SIC3	0.36		0.34	0.31	0.34	0.34
SIC4	0.36	0.43		0.31	0.37	0.37
SIC5	0.36	0.39	0.39		0.31	0.31
SIC6	0.36	0.43	0.46	0.39		0.38
SIC7	0.36	0.43	0.46	0.39	0.47	

Panel C: 20% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.32	0.32	0.32	0.32	0.32
SIC3	0.40		0.37	0.34	0.37	0.37
SIC4	0.40	0.45		0.34	0.39	0.39
SIC5	0.40	0.42	0.42		0.34	0.34
SIC6	0.40	0.45	0.48	0.42		0.40
SIC7	0.40	0.45	0.48	0.42	0.48	

Panel D: 30% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.35	0.35	0.35	0.35	0.35
SIC3	0.44		0.39	0.37	0.39	0.39
SIC4	0.44	0.47		0.37	0.41	0.41
SIC5	0.44	0.45	0.45		0.37	0.37
SIC6	0.44	0.47	0.49	0.45		0.41
SIC7	0.44	0.47	0.49	0.45	0.50	

Note: Tail dependence implied by the Factor skew t-t model of Oh and Patton (2017), using the residuals of the regression for the DJ30 returns on the first FFC factor. The lower (upper) triangular entries correspond to dependence coefficients in the left (right) tail.

Table 6: Lower\ Upper tail dependence coefficients among six industries based on CTI measure

Panel A: 5% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.17	0.15	0.12	0.14	0.09
SIC3	0.20		0.17	0.13	0.24	0.17
SIC4	0.16	0.17		0.13	0.17	0.11
SIC5	0.16	0.14	0.12		0.14	0.10
SIC6	0.15	0.25	0.15	0.11		0.12
SIC7	0.12	0.15	0.10	0.11	0.09	

Panel B: 10% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.16	0.15	0.12	0.14	0.08
SIC3	0.19		0.13	0.12	0.32	0.17
SIC4	0.15	0.16		0.07	0.12	0.08
SIC5	0.15	0.16	0.14		0.12	0.08
SIC6	0.14	0.24	0.12	0.13		0.10
SIC7	0.10	0.17	0.09	0.10	0.09	

Panel C: 20% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.16	0.13	0.12	0.13	0.09
SIC3	0.16		0.12	0.13	0.18	0.14
SIC4	0.13	0.13		0.09	0.12	0.07
SIC5	0.14	0.13	0.11		0.12	0.09
SIC6	0.14	0.20	0.13	0.12		0.09
SIC7	0.10	0.17	0.09	0.09	0.09	

Panel D: 30% quantile

	SIC2	SIC3	SIC4	SIC5	SIC6	SIC7
SIC2		0.14	0.11	0.10	0.10	0.07
SIC3	0.15		0.11	0.12	0.15	0.13
SIC4	0.13	0.12		0.08	0.10	0.06
SIC5	0.13	0.13	0.09		0.12	0.07
SIC6	0.12	0.18	0.11	0.10		0.08
SIC7	0.10	0.16	0.09	0.09	0.10	

Note: Tail dependence implied by CTI measure, using the residuals of the regression for the DJ30 returns on the first FFC factor. The lower (upper) triangular entries correspond to dependence coefficients in the left (right) tail. Refer to the main text for details.